5-5-2015

Bipartite Ramsey Numbers and Zarankiewicz Numbers

Alex F. Collins

MS Thesis:
Bipartite Ramsey Numbers and Zarankiewicz Numbers

Applied and Computational Mathematics
School of Mathematical Sciences
College of Science
Rochester Institute of Technology

Alex F. Collins

Committee Members:

Stanisław Radziszowski, Ph.D. (Advisor)
Department of Computer Science,
Rochester Institute of Technology

Darren Narayan, Ph.D.
School of Mathematical Sciences,
Rochester Institute of Technology

Hossein Shahmohamad, Ph.D.
School of Mathematical Sciences,
Rochester Institute of Technology

5 May 2015

Bipartite Ramsey Numbers and Zarankiewicz Numbers

Alex F. Collins

Abstract

The bipartite Ramsey number $b(m,n)$ is the minimum b such that any 2-coloring of $K_{b,b}$ results in a monochromatic $K_{m,m}$ subgraph in the first color or a monochromatic $K_{n,n}$ subgraph in the second color. The Zarankiewicz number $z(m,n;s,t)$ is the maximum size among $K_{s,t}$-free subgraphs of $K_{m,n}$. In this work, we discuss the intimate relationship between the two numbers as well as propose a new method for bounding the Zarankiewicz numbers. We use the better bounds to improve the upper bound on $b(2,5)$, in addition we improve the lower bound of $b(2,5)$ by construction. The new bounds are shown to be $17 \leq b(2,5) \leq 18$. Additionally, we apply the same methods to the multicolor case $b(2,2,3)$ which has previously not been studied and determine bounds to be $16 \leq b(2,2,3) \leq 23$.

Committee Approval:

Dr. Stanisław Radziszowski Date
Department of Computer Science
Thesis Advisor

Dr. Darren Narayan Date
School of Mathematical Sciences
Committee Member

Dr. Hossein Shahmohamad Date
School of Mathematical Sciences
Committee Member

Dr. Nathan Cahill Date
School of Mathematical Sciences
Director of Graduate Programs

Contents

0 Introduction

Ramsey Theory can be broadly defined as the study of how order emerges from chaos with respect to graph theory. This branch of extremal graph theory asks how large a graph must be in order to gaurantee that it contains a certain subgraph in the graph itself or in its complement.

To cite an overused and somewhat contrived example, suppose that we are throwing a party. We want this party to be either a 'gathering of friends' in which at least four people mutually know each other, or a 'get to know each other party' where there are at least four mutual non-acquaintances, but we don't particularly care which type of party it ends up being. How many people do we need in order to guarantee that we satisfy at least one of the above conditions?

If we state were to state this as a graph theory problem it would go as follows. Define the graph where each person at the party is represented by a vertex. We will add an edge between each vertex if the associated attendees are acquaintances. Then the complement of our graph gives us the non-acquaintances in attendance. Our problem is then equivalent to the Ramsey number $R(4, 4)$ which asks how large does a graph have to be such that it is quaranteed to contain a K_4 subgraph itself (four mutual acquaintances) or in its complement (four mutual non-acquaintances). $R(4, 4)$ is known to be 18.

So we need at least 18 people in order to have one of the two types of parties that we want. Certainly we could have four mutual acquaintances or non-aquaintances at a smaller party, but it is not guaranteed. We could also change our definitions of the two types of parties, and then we would need to use a different Ramsey number to solve our problem.

Solving for Ramsey numbers has proved to be a difficult problem, and $R(4, 4)$ is the largest diagonal case currently known. The next case is $R(5, 5)$ with the current bounds $43 \leq R(5, 5) \leq 49$. The bounds on some small Ramsey numbers have not been improved in 30 or 40 years due to the computational intractability of dealing exhaustively with graphs of the required magnitude.

Bipartite Ramsey numbers are similarly defined for bipartite graphs. These numbers ask how large does a bipartite graph have to be such that it contains a given bipartite subgraph or that its bipartite complement contains another bipartite subgraph. These numbers seem to be at least as difficult as the Ramsey numbers to calculate exactly.

Due to the difficulty of calculating values exactly, many authors have studied the limiting behaviors of these types of numbers or attacked the numbers using probabalistic methods instead. In this work we attempt instead to calculate small bipartite Ramsey numbers exactly, despite the difficulty. While we do not succeed in solving any new numbers, we present improved bounds on $b(2,5)$ and new bounds on the unstudied multicolor number $b(2,2,3)$.

In Section 1 we discuss the level of graph theory knowledge that is assumed and introduce the the terminology that is specific to Ramsey theory or this work. In Section 2 we discuss the work to date on a broad range of topics related to Ramsey theory in general and bipartite Ramsey numbers in particular. In Section 3 we introduce a new bounding method for the Zarankiewicz numbers. The Zarankiewicz numbers are used to provide upper bounds on the bipartite Ramsey numbers, so the implications of the new techniques with regard to $b(2,5)$ are explained in Section 4, which also discusses the improvement of the lower bound of $b(2,5)$ and related work. Section 5 uses the same methods to give bounds for the three color bipartite Ramsey number $b(2,2,3)$, which has not previously been studied as far as we are aware. Detailed explainations of the computational techniques used throughout the project are located in Section 6. Finally, we discuss additional directions and further possible avenues of research in Section 7. Supporting materials, including graphs and Zarankiewicz tables can be found in Appendices B and C respectively.

1 Definitions

A basic knowledge of graph theory will be assumed of the reader. In order to understand the definitions set forth below, the reader should be familiar with the common types of graphs, including complete graphs, bipartite graphs, and cycles, as well as some topics in graph theory such as edge colorings and complementation. Nonstandard definitions and definitions specific to Ramsey theory are listed below. These definitions are consistent with those given in Bollobás' survey [Bol95].

- **Bipartite Complement:** if G is a bipartite graph with partitions X and Y, then the bipartite complement of G (which we will denote $\ddot{G}$) is a bipartite graph whose vertex set is the vertex set of G and whose edge set consists of all of the edges between X and Y which do not appear in G.

- **Ramsey Number:** $R(m, n)$ is the minimum order of a complete graph such that any 2-coloring of the edges must result in either a complete graph of order m in the first color or a complete graph of order n in the second color. A simple example is given in Figure 1.0. The multicolor Ramsey number $R(n_1, \ldots, n_c)$ is the obvious generalization with c colors.

- **Bipartite Ramsey Number:** $b(m, n)$ is the minimum b such that any 2-coloring of the edges of $K_{b,b}$ must result in either a subgraph of $K_{m,m}$ in the first color of a subgraph of $K_{n,n}$ in the second color. A simple example is given in Figure 1.1. The multicolor bipartite Ramsey number $b(n_1, \cdots, n_c)$ is the obvious generalization with c colors.

- **Zarankiewicz Number:** $z(m, n; s, t)$ is the maximum size among $K_{s,t}$-free subgraphs of $K_{m,n}$. If $m = n$ and $s = t$, we write $z(m; s)$.

- **Witness Graph:** A graph G is a witness graph for the lower bound on a Zarankiewicz number $z_1 \leq z(m, n; s, t)$ if G is a $K_{m,n}$ subgraph with z_1 edges that does not contain any $K_{s,t}$ subgraphs. G proves the lower bound by demonstrating that such a graph exists. Similarly, G is a witness graph to the lower bound on a bipartite Ramsey number $b_1 \leq b(m, n)$ if G is a K_{b_1,b_1} subgraph that does not contain a $K_{m,m}$ subgraph and does not contain a $K_{n,n}$ subgraph in its bipartite complement. Again, G demonstrates that such a graph exists, thus proving the lower bound. A similar definition can be stated for the witness graph of the lower bound on a Ramsey number $R(m, n)$.

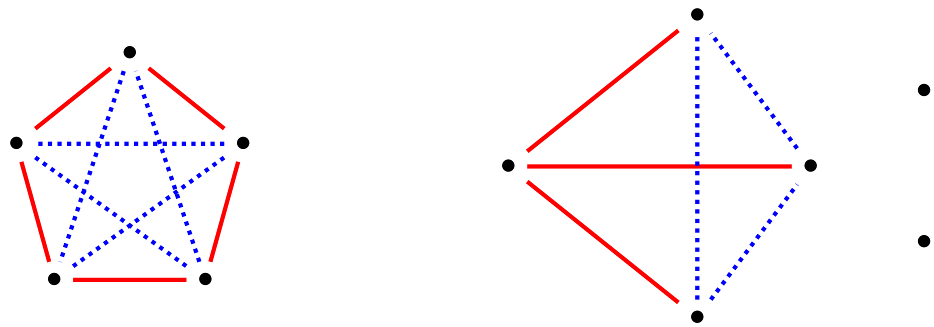

Figure 1.0: The image on the left is a 2-coloring of the edges of K_5 without monochromatic triangles. Thus $R(3,3) > 5$. The image on the right shows that the edges of K_6 cannot be colored with two colors while avoiding monochromatic triangles (since at least three edges from a single vertex v must have the same color by the pigeon-hole principle, without loss of generality, let these edges be red, but then the edges between the vertices adjacent to v by a red edges must be blue to avoid a red triangle, but this gives rise to a blue triangle). Thus $R(3,3) = 6$.

2 History and Background

2.0 Ramsey's Theorem

In his 1930 paper [Rms30] on formal logic, Ramsey proved that if the r-combinations of an infinite class Γ are colored in c distinct colors, then there exists a subclass $\Delta \subset \Gamma$ such that all of the r-combinations of Δ are the same color. For $r = 2$, this is equivalent to saying that an infinite complete graph whose edges are colored in c colors contains an infinite monochromatic complete subgraph. For $r > 2$, an equivalent statement can be made in terms of hypergraph edge-colorings [Rms30].

He also proved a finite case of this theorem, which simply stated means that the Ramsey number $R(n,n)$ is finite [Rms30]. Clearly this implies that $R(m,n)$ is finite in general. This gave rise to the study of the Ramsey numbers.

The classical proof of Ramsey's theorem is given in Gasarch's survey along with a summary of related results [Gsr98].

2.1 A Problem of Zarankiewicz

In 1951, Zarankiewicz asked how many ones could be fit into an $n \times n$ matrix while avoiding a 3×3 submatrix of ones (for $n = 4, 5, 6$) [Zkz51]. This can be stated as a graph theory problem as follows: what is the maximum number of edges in a subgraph of $K_{n,n}$ that does not contain a $K_{3,3}$ subgraph. The problem was soon generalized

Figure 1.1: A 2-coloring of $K_{4,4}$ without monochromatic 4-cycles (left), showing that $b(2,2) > 4$; the degree sequence $3,3,3,2,2$ cannot be used in a 2-coloring of $K_{5,5}$ (right), it is easy to see that other degree sequences also cannot be used. Thus $b(2,2) = 5$.

for graphs and subgraphs of arbitrary sizes. The resulting numbers became to be known as the Zarankiewicz numbers, and a concise definition of the meaning of the generalized case $z(m,n;s,t)$ is given above.

By 1969, Guy had compiled tables of exact values for Zarankiewicz numbers with small parameters, all of which had been computed by hand using combinatorial argument and other techniques. Guy also included a list of techniques and results that had been discovered at that time, as well as a number of applications to other areas of mathematics (such as the graph theory connection noted above) [Guy69].

While Guy's work is extraordinarily useful and contains many useful insights, there is at least one error in his tables of small Zarankiewicz numbers. It should be noted that the error was in $z(15;2)$, the largest diagonal case attempted in the C_4-free table, and his value was only off by one. This discrepancy is significant because at least one author (Héger) has reported the erroneous value as well as further values based on it without having checked Guy's tables sufficiently [Hgr13]. With the advent of computational techniques, we should be able to check and extend Guy's tables.

A number of other authors have studied Zarankiewicz numbers, including Kövári, Sós, and Turán [KvrET54], Balbuena et al. [BalbET07], Dutta and Radhakrishnan [DutRdh12], Nikiforov [Nkr10], and Reiman [Rmn58]. A good summary of progress can be found in Bollobás [Bol95].

2.2 Relating the Problems

In 1975, Beineke and Schwenk defined the bipartite Ramsey number $r(m,n)$ to be the smallest number p such that any 2-coloring of the edges of $K_{p,p}$ contains a monochromatic $K_{m,n}$ subgraph [BnkSwk75]. Note that this is different from the bipartite Ramsey number $b(m,n)$ which we are interested in (and which is defined above). However,

	2	3	4	5	6
2	5	9	14	≤ 19	≤ 25
3		17	≤ 29	≤ 41	≤ 56
4			≤ 48	≤ 72	≤ 101
5				≤ 115	≤ 168

Table 2.0: The upper bounds for small cases of the bipartite Ramsey numbers reported by Goddard, Henning, and Oellermann. Several of these are their own improvements. Although they did not calculate lower bounds for all cases, they did report a lower bound of 16 for $b(2,5)$ [GdrET00].

it is clear that the values coincide in the diagonal case (i.e. $b(n,n) = r(n,n)$).

While Irving worked primarily with the definition by Beineke and Schwenk, his work from 1978 [Irv78] nonetheless proved the following bound on bipartite Ramsey numbers as we know them:

$$z(b;m) + z(b;n) < b^2 \implies b(m,n) \leq b.$$

Thus the two problems are related in that Zarankiewicz numbers provide an upper bound on the bipartite Ramsey numbers.

2.3 Recent Developments

In 2000, Goddard, Henning, and Oellermann used a linear programming algorithm based on a lemma from Irving's paper to bound small Zarankiewicz numbers. They used these bounds in turn to bound small bipartite Ramsey numbers. The results that they reported are repeated in Table 2.0. They also found a lower bound for $b(2,5) > 15$ [GdrET00].

In 2013, Dybizbański, Dzido, and Radziszowski proved theorems giving exact results for Zarankiewicz numbers with $s = t = 2$ and $m = n = k^2 + k + 1 - h$ for $0 \leq h \leq 3$ and k a prime power. These results are useful for bounding bipartite Ramsey numbers with at least one parameter being 2 since there is no bound on the Zarankiewicz parameters, just some restriction on k. They also used these results to find new multicolor bipartite Ramsey numbers (to be introduced and discussed below) [DybET13].

Additional notable work includes that of Hattingh and Henning [HttHnn98], Lazebnik and Mubayi [LzbMub02], and Conlon [Cnl08]. The survey by Fürdei and Simonovits [FrdSim13] contains a great deal of the recent developments on Zarankiewicz numbers and bipartite Ramsey numbers.

2.4 Multicolorings

Natural generalizations of Ramsey numbers and bipartite Ramsey numbers exist for colorings with more than two colors. In the bipartite case only two such values are known; $b(2,2,2) = 11$ [Exo91] and $b(2,2,2,2) = 19$ [DybET13] (denoted $b_3(2), b_4(2)$ and $b_c(2)$ for c colors in general). It is also known that $26 \leq b_5(2) \leq 28$ [DybET13]. Although unbalanced cases have not previously been studied, the methods proposed in this work can be used to bound them.

Fenner et al. [FnrET10] and Steinbach and Posthoff [StbPhf12] have also studied multi-color cases using the grid coloring approaches presented below. Lazebnik and Woldar [LzbWld00] have also studied multicolor cases.

2.5 Alternative Approaches

2.5.0 Purely Computational

Werner has proposed an algorithmic approach to determining the exact value of $\mathbf{maxrf}(m,n)$ (maximal rectangle-free grids, analogous to $z(m,n;2,2)$) [Wrn12]. However the algorithm is too slow to check the relevant regions of Guy's tables in any reasonable ammoumt of time. Barring an exponential speed-up, it is unlikely that this approach will prove to be fruitful.

McKay's *nauty* software package includes graph generation scripts which use canonical labellings to generate non-isomorphic graphs with specified parameters [MKy13]. These programs are incredibly useful for finding witness graphs to lower bound improvements for small cases of bipartite Ramsey numbers.

2.5.1 Grid Colorings

A separate group of researchers has been working independently on grid coloring problems, in which they attempt to color $m \times n$ grids with c colors while avoiding monochromatic rectangles (their term for 2×2 subgrids). The connections to Zarankiewicz numbers and multicolor bipartite Ramsey numbers are clear, but until recently these researchers had been unaware of the theoretical background.

The authors most active in this branch include Fenner et al. [FnrET10] and Steinbach and Posthoff [StbPhf12].

2.5.2 Projective Planes

A projective plane is a geometric structure consisting of lines and points such that every pair of lines intersect at exactly one point and every pair of points are both incident to exactly one line. These structures are useful because their bipartite graph representations (where one partition consists of lines and the other of points, with edges representing incidence) cannot contain C_4 subgraphs.

Many authors have used these structures in order to develop constructions for upper bounds on Zarankiewicz numbers and lower bounds on bipartite Ramsey numbers avoiding $K_{2,2}$. Some of the most notable developments include those by Alm and Manske [AlmMan12], Dybizbański, Dzido, and Radziszowski [DybET13], Erdős, Rényi, and Sós [EReSos66], and Parsons [Prs76].

2.5.3 Asymptotics

Rather than focusing on small and potentially computable Zarankiewicz numbers and bipartite Ramsey, some researchers have instead chosen to study the limiting cases. The most significant known asymptotic bounds are due to Kővari, Sós, and Turán, who showed that

$$\lim_{n\to\infty} \frac{z(n;2)}{n^{3/2}} = 1. \text{ [KvrET54]}$$

Other asymptotics work have been derived by Ball and Pepe [BllPpe12], Brown [Bwn66], and Lin, and Li [LinLi09].

	3	4	5	6	7	8	9	10	11	12	13
3	8	10	12	14	16	18	20	22	24	26	28
4		13	16	18	21	24	26	28	30	32	34
5			20	22	25	28	30	33	36	38	41
6				26	29	32	36	39	42	45	48
7					33	37	40	44	47	50	53
8						42	45	50	53	57	60
9							49	54	59	64	
10								60			

Table 3.0: Guy's table of Zarankiewicz numbers $z(m, n; 3)$ [Guy69]. Note that in Guy's original table, he defined Zarankiewicz numbers as the minimum number of edges in a $K_{m,n}$ subgraph to guarantee a $K_{s,t}$ subgraph, which is equivalent, but all of his numbers are increased by one. The numbers here are according to the definition given at the beginning of this paper. Note also that Guy's original table extends out 10 more columns to the right. A duplicate of this table appears in Appendix C.

3 Zarankiewicz Numbers

3.0 New Upper Bounding Theorem

Theorem:
If $z(m, n; s) = z_1, (m > n)$ and $z(m, n + 1; s) = z_2$, then

$$z_2 - z_1 \leq \min\left\{ \left\lfloor \frac{z_1}{n} \right\rfloor, \left\lfloor \frac{z_2}{n+1} \right\rfloor \right\}.$$

Similarly, if $z(m, n; s) = z_1, (m \geq n)$ and $z(m + 1, n; s) = z_2$, then

$$z_2 - z_1 \leq \min\left\{ \left\lfloor \frac{z_1}{m} \right\rfloor, \left\lfloor \frac{z_2}{m+1} \right\rfloor \right\}.$$

Proof:
We prove the first statement as follows:

Let $z_2 = z_1 + \left\lfloor \frac{z_1}{n} \right\rfloor + 1$. Let G be a subgraph of $K_{m,n+1}$ with z_2 edges that avoids $K_{s,s}$

subgraphs. Then G has a vertex v in its $(n+1)$-vertex partition with degree at most

$$\begin{aligned}
\deg(v) &\leq \left\lfloor \frac{\left\lfloor \frac{z_1}{n} \right\rfloor + z_1 + 1}{n+1} \right\rfloor \\
&= \left\lfloor \frac{\left\lfloor \frac{(n+1)z_1}{n} \right\rfloor + 1}{n+1} \right\rfloor \\
&= \left\lfloor \frac{(n+1)\left\lfloor \frac{z_1}{n} \right\rfloor + \left\lfloor (n+1)\left(\frac{z_1}{n} - \left\lfloor \frac{z_1}{n} \right\rfloor\right) \right\rfloor + 1}{n+1} \right\rfloor \\
&\leq \left\lfloor \frac{(n+1)\left\lfloor \frac{z_1}{n} \right\rfloor + \left\lfloor (n+1)\left(\frac{cn-1}{n} - \left\lfloor \frac{cn-1}{n} \right\rfloor\right) \right\rfloor + 1}{n+1} \right\rfloor \\
&= \left\lfloor \frac{(n+1)\left\lfloor \frac{z_1}{n} \right\rfloor + (n-1) + 1}{n+1} \right\rfloor \\
&= \left\lfloor \left\lfloor \frac{z_1}{n} \right\rfloor + \frac{n}{n+1} \right\rfloor \\
&= \left\lfloor \frac{z_1}{n} \right\rfloor
\end{aligned}$$

But then $G - v$ is a $K_{m,n}$ subgraph with $z_1 + 1$ edges that avoids $K_{s,s}$ subgraphs, contradicting our assumption.

Alternatively, let $z_1 = z_2 - \left\lfloor \frac{z_2}{n+1} \right\rfloor - 1$. Let H be a subgraph of $K_{m,n+1}$ with z_2 edges that avoids $K_{s,s}$ subgraphs. Then H has a vertex u in the $(n+1)$-partition with degree at most $\left\lfloor \frac{z_2}{n+1} \right\rfloor$. But then $H - u$ is a $K_{m,n}$ subgraph with $z_1 + 1$ edges that avoids a $K_{s,s}$ subgraph, leading to another contradiction.

The second statement follows by a similar argument. QED.

3.1 Application of Bounding Theorem

Knowing smaller Zarankiewicz numbers (or even just upper bounds on such) can now allow us to find nontrivial upper bounds on larger Zarankiewicz numbers. The obvious approach uses dynamic programming to go row by row through a table of Zarankiewicz numbers bounding those that are not already known using the bounds from the numbers immediately above and to the left. In this way we can build up to higher and more interesting numbers that can be used to bound bipartite Ramsey numbers using Irving's result.

	6	7	8	9	10	11	12	13	14	15
6	**26**	**29**	**32**	**36**	39	42	45	48	51	54
7		33	37	41	45	49	52	56	59	63
8			42	46	51	56	59	63	67	71
9				51	56	61	66	70	75	79
10					62	67	73	77	82	87
11						73	79	84	90	95
12							86	91	98	103
13								98	105	111
14									113	119
15										127

Table 3.1: An extended table of the Zarankiewicz numbers $z(m,n;3)$. While related to Guy's table (see Table C.1) this does not rely on Guy for accuracy; all exact values (shown in bold) were independently checked using nauty. Upper bounds are shown are displayed in plain font and were generated using the theorem in Section 3.0. A duplicate of this table appears in Appendix C.

Table 3.1 shows exactly this extension for the $z(m,n;3)$ table. Note that while this table includes some of the data from Guy's table, only the values that were checked using nauty were used to compute the new bounds. Similar tables for $s = 2, 4, 5$, and 6 are included in Appendix C, in addition to a duplicate of Table 3.1 for convenience.

3.2 Error Estimation for Upper Bound

An important question to ask about these new bounds are how tight they are. Using theoretical results [DybET13] we can compute $z(28;2) = 156$. The largest smaller known Zarankiewicz number for quadrilaterals is $z(21;2) = 105$ which comes from the same paper. This suggests an excellent experiment to determine the tightness of the new bounding technique.

We ran the dynamic programming extension code to generate a table of Zarankiewicz bounds avoiding quadrilaterals up to $z(30;2)$ to see how far off the bounds would be. Part of the results are shown in Table C.3, but the section relevant to this experiment is shown in Table 3.2. Clearly this demonstrates that the new bounding technique is not tight; the bound given by the program is $z(28;2) \leq 165$.

We suspect that this error snowballs the further we try to extend the bounds. In addition, we believe that this snowballing error is the reason why these bounds do not produce improved bounds for larger bipartite Ramsey numbers such as $b(2,6)$

	27	28	29	30
27	≤ 155	≤ 160	≤ 165	≤ 170
28		156	≤ 161	≤ 166
29			165	170
30				175

Table 3.2: The section of the $z(m, n; 2)$ extension table relevant to an experiment to determine the approximate error produced by the bounding technique. Exact values on the diagonal were computed using theoretical results [DybET13]. From those known values it can be seen by the theorem in this section that $z(29, 30; 2) - z(29; 2) \leq 5$ and $z(30; 2) - z(29, 30; 2) \leq 5$, thus giving the equality $z(29, 30; 2) = 170$. Note that a similar reasoning was attempted for other values but was unsuccessful. The other bounds were determined using the dynamic programming approach discussed in this section.

and $b(3, 4)$ (see Section 4.1).

3.3 Computational Details

The code used to extend Zarankiewicz tables is among the most simple of all the code written for this project. For each table the known values had to be hard-coded into an array, with all other elements being set to zero. Then we would iterate through the above-diagonal elements of the array, with any zero elements being over-written by the new bounds computed from the elements immediately above and to the left. Obviously diagonal bounds were computed only from the element immediately above them.

4 Bipartite Ramsey Numbers

4.0 $b(2,5)$

Recall from Section 2.3 the bounds $16 \leq b(2,5) \leq 19$.

Theorem:

$$17 \leq b(2,5) \leq 18.$$

Proof:
The lower bound is shown by constructing a witness graph. Namely a 2-coloring of $K_{16,16}$ that does not contain a $K_{2,2}$ in the first color or a $K_{5,5}$ in the second color. Such a graph was found using nauty. The first color is shown in Figure B.0 with its bipartite adjacency matrix in Table B.0. Note that Figure B.0 shows only the colored edges that avoid $K_{2,2}$ (the four colors are to show the structure of the graph), the bipartite complement avoids $K_{5,5}$. So $17 \leq b(2,5)$.

The upper bound is improved by a theoretical argument using Zarankiewicz numbers. In order for a 2-coloring of $K_{18,18}$ avoiding $K_{2,2}$ in the first color and $K_{5,5}$ in the second color to exist, we can have at most $z(18;2) = 81$ [DybET13] edges in the first color and at most $z(18;5) \leq 243$ (see Table C.6) edges in that second color. Thus we have a total of at most $81 + 243 = 324 = 18^2$, so a simple application of Irving's result does not suffice. Instead we must use the additional fact that the witness for $z(18;2) = 81$ is unique [DybET13] and that its bipartite complement contains a $K_{5,5}$ subgraph. Thus if we are to avoid both forbidden subgraphs, we can only have at most 80 edges in the first color, and at most $80+243 = 323 < 18^2$ edges in the whole graph. Thus, by Irving, $b(2,5) \leq 18$. QED.

4.1 Generalization of Upper Bounding Technique

While the upper bounding method in the above proof is theoretically generalizable to any bipartite Ramsey problem, the bounding of the Zarankiewicz numbers is not tight, and the small errors appear to snowball (see further discussion in Section 3). It is because of this snowballing error that we make the following conjecture.

Conjecture: $b(2,5) = 17$.

The current bound $z(17,17;5) \leq 221$ need only be reduced to $z(17,17;5) \leq 214$ in order to prove the above conjecture by Irving's technique. Given the error accumulation discovered in bounding calculations with a similar number of iterations (see

n	$z(n;2)$	$z(n;6)$	$z(n;2)+z(n;6)$	n^2
18	**81**	264	345	324
19	**88**	293	381	361
20	**96**	324	420	400
21	**105**	357	462	441
22	114	387	501	484
23	123	422	545	529
24	130	459	589	576
25	138	494	632	625

Table 4.0: The current bounds on $z(n;2)$ and $z(n;6)$ for the relevant n necessary to improve the bounds on $b(2,6)$. Exact values of $z(n;2)$ are shown in bold and are from [DybET13]. All other Zarankiewicz numbers listed are upper bounds only and are obtained from the Zarankiewicz extension technique discussed in Section 3. The full tables for these numbers can be found in Appendix C.

Section 3), it would not be surprising if this bound could be reduced as required.

Unfortunately, this same accumulation of errors prevents us from improving the bound on other small bipartite Ramsey numbers. The next two smallest such numbers are $b(2,6) \leq 25$ and $b(3,4) \leq 29$ [GdrET00]. Table 4.0 shows the current bounds on $z(n;2)$ and $z(n;6)$ for the n that would be relevant to improve the bounds of $b(2,6)$ and why they are not sufficient. Table 4.1 shows the same for $b(3,4)$.

4.2 Computational Details

As mentioned above, the lower bound was improved using nauty. To keep the computation feasible, the following restrictions were used:

- Generate bipartite subgraphs of $K_{16,16}$ that avoid $K_{2,2}$ subgraphs,
- that have between 64 and 67 edges,
- and that have minimum degree 4.

The maximum size of the graph was chosen because $z(16;2) = 67$ [DybET13], the minimum size was chosen to enforce the minimum degree. A smaller minimum degree was attempted, but did not appear to be computationally feasible.

nauty generated 32 graphs for the above computation. Somewhat surprisingly the only one that avoided $K_{5,5}$ in its bipartite complement had only 64 edges (and thus had the most edges in its bipartite complement). We attempted to extend this graph to a $K_{17,17}$ subgraph in order to improve the lower bound further (see Section 6.1 for

n	$z(n;3)$	$z(n;4)$	$z(n;3)+z(n;4)$	n^2
18	177	220	397	324
19	196	242	436	361
20	215	267	482	400
21	236	294	530	441
22	257	320	577	484
23	280	349	629	529
24	301	377	678	576
25	326	408	734	625
26	348	439	787	676
27	371	471	842	729
28	398	506	904	784
29	422	538	960	841

Table 4.1: The current bounds on $z(n;3)$ and $z(n;4)$ for the relevant n necessary to improve the bounds on $b(3,4)$. All of these numbers are upper bounds only; none are known to be exact. These numbers go well beyond the tables in Appendix C, but the methodology remains the same.

method) but this failed.

n	$z(n;2)$	$z(n;3)$	$2z(n;2)+z(n;3)$	n^2
18	**81**	177	339	324
19	**88**	196	372	361
20	**96**	215	407	400
21	**105**	236	446	441
22	114	257	485	484
23	123	280	526	529

Table 5.0: The current bounds on $z(n;2)$ and $z(n;3)$ for the relevant n to prove the bound $b(2,2,3) \leq 23$. Exact values of $z(n;2)$ are shown in bold, all other Zarankiewicz numbers listed are current upper bounds. Note that there was no known prior bound for $b(2,2,3)$.

5 Multicolor Bipartite Ramsey Numbers

5.0 $b(2,2,3)$

Theorem:

$$16 \leq b(2,2,3) \leq 23.$$

Proof:
Irving's result can be generalized to k colors as follows:

$$\sum_{i=1}^{k} z(b;n_i) < b^2 \implies b(n_1,n_2,\cdots,n_k) \leq b.$$

With Irving's original argument easily sufficing for a proof. Thus we can improve the upper bounds on multicolor Zarankiewicz numbers in the same way that was used for the two color case. Table 5.0 demonstrates the use of the generalized Irving result to prove the stated upper bound.

The lower bound must be shown by construction of a witness graph. There are 26 unique witnesses shown in Table B.1. These are trinary sequence representations of cyclic $K_{15,15}$ graphs which do note have C_4s in the first two colors or $K_{3,3}$s in the third color. They can be converted from string representation to bipartite adjacency matrices by allowing each rotation of the sequence to be one row of the matrix. An example for the first listed witness in shown in Figure 5.0. None of these witnesses were able to be extended. QED.

0 0 1 0 1 2 2 0 2 2 2 2 1 1 2
↳ 0 0 1 0 1 2 2 0 2 2 2 2 1 1 2
↳ 0 0 1 0 1 2 2 0 2 2 2 2 1 1 2
↳ etc.

Wrapping sequences around

$$\begin{pmatrix}
0&0&1&0&1&2&2&0&2&2&2&2&1&1&2\\
2&0&0&1&0&1&2&2&0&2&2&2&2&1&1\\
1&2&0&0&1&0&1&2&2&0&2&2&2&2&1\\
1&1&2&0&0&1&0&1&2&2&0&2&2&2&2\\
2&1&1&2&0&0&1&0&1&2&2&0&2&2&2\\
2&2&1&1&2&0&0&1&0&1&2&2&0&2&2\\
2&2&2&1&1&2&0&0&1&0&1&2&2&0&2\\
2&2&2&2&1&1&2&0&0&1&0&1&2&2&0\\
0&2&2&2&2&1&1&2&0&0&1&0&1&2&2\\
2&0&2&2&2&2&1&1&2&0&0&1&0&1&2\\
2&2&0&2&2&2&2&1&1&2&0&0&1&0&1\\
1&2&2&0&2&2&2&2&1&1&2&0&0&1&0\\
0&1&2&2&0&2&2&2&2&1&1&2&0&0&1\\
1&0&1&2&2&0&2&2&2&2&1&1&2&0&0\\
0&1&0&1&2&2&0&2&2&2&2&1&1&2&0
\end{pmatrix}$$

Figure 5.0: An example showing how to convert one of the 26 trinary sequences listed in Table B.1 into the associated bipartite adjacency matrix by using each of the possible rotations as one line of the matrix.

5.1 Computational Details

The witness graphs for the lower bound were generated using the method discussed in Section 6.0.1. As discussed in the example in that section, the parameters used were $(4, 4, 7)$ and $(2, 2, 3)$, meaning that each vertex would be incident to four edges in each of the first two colors (which avoid $K_{2,2}$) and seven edges in the third (which avoids $K_{3,3}$).

The parameters $(4, 4, 7)$ were chosen after some experimentation because they would result in 60 edges of each of the first two colors and $z(15; 2) = 61$ while the edges in the third color would also be less than the current bound on $z(15; 3) \leq 127$. By the same logic, the parameters $(4, 4, 8)$ were attempted in hopes that a suitable $K_{16,16}$ coloring could be found to improve the bound further, but none were generated.

It seems highly unlikely that this method would produce the best possible witness, but a full exhaustive search was computationally infeasible. This method at least produced a nontrivial lower bound to use as a starting point for further searches.

6 Computational Details

6.0 Generating Graphs

6.0.0 nauty

The genbg function from nauty was used extensively to generate exhaustive witness lists for Zarankiewicz numbers. For example, to generate such a list for $z(m,n;s,t) = z_1$ witnesses (knowing that $z(m,n;s,t) \leq z_1$), the following process was used:

- Run genbg with the appropriate parameters to generate all $K_{m,n}$ subgraphs with exactly z_1 edges. An example call of genbg is provided at the end of this section.
- Check each generated graph to see if it contains a $K_{s,t}$ subgraph (see Subgraph Checking, below).
- Save the generated graphs that do not contain the forbidden subgraphs, this is an exhaustive list of witnesses.
- If no such graphs exist, repeat process for using $z_1 - 1$ edges.

This method is computationally infeasible for graphs with more than ≈ 20 vertices. So to get witnesses for larger Zarankiewicz numbers, we must extend the witnesses of prior Zarankiewicz numbers, using the methods discussed in Extending Graphs below.

Note that a parameter exists to enforce the generated graphs to be C_4-free, so in the case of $s = t = 2$ we can skip the subgraph checking step.

Example genbg call:
The call >genbg -u -l -Z1 -d4:4 -D7:7 16 16 64:67 produces the following output:

>A genbg n=16+16 e=64:67 d=4:4 D=7:7 Z1
>Z 32 graphs generated in 175.26 sec

This was the command used to generate the witness graph that improved the lower bound of $b(2,5)$.

Note that the '-u' option suppresses some of the output so that none of the generated graphs are displayed. To see all of the graphs this option should be removed. To store the graphs in a file simply add the filename to the end of the command.

6.0.1 Cyclic graphs

In order to generate a cyclic graph coloring, we must first determine how many edges of each color are incident to each vertex. This was typically done using educated guesses based on the Zarankiewicz numbers. See Section 5 for a concrete example.

Then the following algorithm will generate all cyclic graph colorings with the given parameters that avoid the given complete bipartite subgraphs. The parameters needed are:

- $(n_0, n_1, \ldots, n_{k-1})$: An array representing the number of edges incident to each vertex for each color. For example $(4, 4, 7)$ indicates that there are to be fifteen vertices in each partition, with each vertex having four incident edges of the first color, four of the second, and seven of the third.
- $(a_0, a_1, \ldots, a_{k-1})$: An array of the same size representing the subgraphs to avoid in each color. To continue the above example $(2, 2, 3)$ would indicate that the first and second color must avoid $K_{2,2}$ subgraphs, while the third must avoid $K_{3,3}$ subgraphs. These were the parameters used to generate $15 < b(2, 2, 3)$ cyclic witness graphs as discussed in Section 5.

The algorithm is as follows:

- Generate all k-nary sequences with n_0 elements being 0, n_1 elements being 1, etc.
- For each such sequence, generate the associated cyclic graph coloring by rotating the sequence to form the rows of the bipartite adjacency matrix.
- Check to determine whether each such graph contains the forbidden subgraphs (see below). Return the associated sequences for those that do not.

This method is obviously not exhaustive, nor is it likely to find the exact lower bound. However it can be useful for finding a non-trivial lower bound and the resulting graphs can sometimes be extended to improve the lower bound further (see below).

6.1 Extending Graphs

Graph extension is extremely useful for improving lower bounds. For example, if we have a graph G witnessing that $b(2, 5) > 16$ (such as the one shown in Figure B.0), we can attempt to find $K_{17,17}$ colorings based off of G which improve the lower bound of $b(2, 5)$ further. We do this by finding all $K_{17,17}$ 2-colorings which have G as an induced subgraph and then checking these new graphs for the forbidden subgraphs. Note that the graph in Figure B.0 cannot be extended to further improve the lower bound of $b(2, 5)$. Furthermore, if we have an exhaustive list of graphs witnessing

$b(2,5) > 16$ (we do not) and none of them were extendable to $K_{17,17}$ colorings that avoid the forbidden subgraphs, then $b(2,5)$ would have to be equal to 17.

The algorithm for extending bipartite graphs is as follows:

- Take a binary bipartite adjacency $n \times n$ matrix as input
- Generate all binary binary sequences of length n, for each such sequence s:
  - Append s to a copy of the matrix as a new column
  - Check whether the new matrix contains a forbidden subgraph, store any that do not
- For each new matrix that did not contain the forbidden subgraphs, repeat the process, this time adding a new row consisting of all the possible $n+1$ length binary sequences
- Any that still do not contain the forbidden subgraphs are witnesses to a new and improved lower bound.

Some notes:

- The algorithm can be modified in the obvious way to extend multi-color graphs as well.
- Since the number of binary sequences grows exponentially with respect to the length, this method can only be used for sufficiently small n.

6.2 Subgraph Checking

Subgraph checking is one of the most important tools, as we need to use it every time we check for a forbidden subgraph.

The algorithm will be easiest to explain and understand through an example. Suppose that we want to check whether a $K_{16,16}$ subgraph contains a $K_{2,2}$. Let M be the bipartite adjacency matrix representing our graph. We would pass $(M, 2, 2)$ as arguments to the algorithm. It will then generate all binary strings of length 16 containing exactly two ones. For each such string, it will extract the two columns cooresponding to the ones and will perform a bitwise AND operation on the two columns. It will count up all of the elements in the result that are still non-zero, if there are at least two, then it will return 'True' indicating that at least one $K_{2,2}$ was found. If the algorithm gets through all such binary sequences without finding any two columns which intersect in at least two elements, then it will return 'False' indicating that M is $K_{2,2}$-free.

7 Further Directions

This section contains a detailed list of approaches to the various problems addressed in this paper that did not pan out due to computational limits and/or time constraints. Many of them deserve further study. Additionally, a list of concrete possibilities for future research is given at the end.

7.0 Lower Bounding Zarankiewicz Numbers

Little success was had in the attempts to find non-trivial lower bounds for Zarankiewicz numbers (based on other Zarankiewicz numbers, as was done for upper bounds). The trivial bound is as follows, if $z(m,n;s) \geq z_1$ then both $z(m+1,n;s)$ and $z(m,n+1;s)$ are at least $z_1 + s - 1$. If a more interesting theoretical lower bound could be determined, then we might have more success in finding exact values of Zarankiewicz numbers further down the tables, which could in turn enable us to push down the upper bounds of more bipartite Ramsey numbers.

By the theorem given in Section 3 if we have $z(m,n-1;s) = z_1$ and $z(m-1,n;s) = z_2$ then

$$z(m,n;s) \leq z_3 = \min\left\{z_1 + \left\lfloor \frac{z_1}{n-1} \right\rfloor, z_2 + \left\lfloor \frac{z_2}{m-1} \right\rfloor\right\}.$$

It was noticed during the process of extending the Zarankiewicz tables that $z_3 - 1 \leq z(m,n;s) \leq z_3$ in all of the (admittedly few) cases that were computable. It is probably worth investigating whether there exists a bound on how far off the theorem can be, and if such a bound exists whether it is constant, linear, quadratic etc. in any of the terms (z_k, m, n or s).

There is a result similar to the theorem given in Sections 3 for providing lower bounds for Zarankiewicz numbers. It states that if $z(m+1,n;s) \geq z_1$ and $z(m,n+1;s) \geq z_2$ then

$$z(m,n;s) \geq z_3 = \max\left\{z_1 - \left\lceil \frac{z_1}{m+1} \right\rceil, z_2 - \left\lceil \frac{z_2}{n+1} \right\rceil\right\}.$$

The proof is obvious, from a witness graph we can remove a vertex of maximal degree from either partition to form a witness graph for the smaller desired Zarankiewicz number. Unfortunately this is not generally useful since we rarely know $z(m+1,n;s)$ or $z(m,n+1;s)$ without first knowing $z(m,n;s)$. We do not expect this result to be useful, nor do we suggest further study on this front. But it is worth noting this result in a discussion about lower bounding theorems for Zarankiewicz numbers.

One additional approach was attempted which did not yeild any results before it had to be dropped due to time constraints. The best way to explain this method is through an example. It goes as follows: knowing that $z(13,17;2) = 59$ and $z(13,18;2) \leq 61$ (see Tables C.3). In the most equitable distribution of edges to vertices in the larger partition, there are eight vertices of degree 4 and nine of degree 3. This covers

$$8 \times \binom{4}{2} + 9 \times \binom{3}{2} = 75 < \binom{13}{2} = 78$$

of the vertex pairs in the small partition. Then there a clearly pairs of vertices which are currently at distance greater than 2 in the smaller partition and a new vertex can be added to the larger partition with an edge to two of these vertices producing a graph with exactly 61 edges.

The problem is that no such witness can exist (otherwise we could add a vertex of degree 3 to the larger partition by a similar argument, but this violates out upper bound). Furthermore, what if the only witnesses that exist have a degree sequence for the larger partition that consists of eleven vertices of degree 4, three of degree 3, and three of degree 2? Then the same computation as done above yeilds

$$11 \times \binom{4}{2} + 3 \times \binom{3}{2} + 3 \times \binom{2}{2} = \binom{13}{2}.$$

Meaning that there as no pairs of vertices in the smaller partition which we could add a new vertex connecting in the larger partition.

In general this method does not work because we need to know something about the degree sequences of the partitions. As such we were unable to derive a general result in the time allotted. It may be possible to prove something about the degree sequences required of a witness graph that can be used to prove a general result. This approach merits additional investigation.

7.1 Exhaustive Search to Solve $b(2,5)$

Here the goal is to generate all $K_{17,17}$ subgraphs G with $68 \leq E(G) \leq 74$ ($17^2 - z(17;5) \geq 68$, $z(17;2) = 74$; see Table C.6 and Table C.3 respectively). Since nauty can only generate graphs with up to 32 vertices, we cannot generate these graphs directly. Instead we must generate all $K_{16,16}$ subgraphs with between 60 and 67 edges (inclusive; bounds obtained in the same manor as above) and extend them to $K_{17,17}$ subgraphs by adding a vertex to each partition and some number of edges from the new vertices.

For obvious reasons, it would be computationally infeasible to exhaustively find all of these subgraphs without sufficiently stringent limiting conditions to reduce the size of the necessary search. The most obvious such conditions are bounds on the degrees of the vertices (this has the additional benefit of being extremely easy to implement using nauty). Two types of arguments can be used to give an upper bound on the degrees of the vertices.

The first argument is a simple one using off-diagonal Zarankiewicz numbers. For example, suppose in the 60 edge case that we have a vertex of degree 5. This vertex has 11 non-edges, which implies that the $K_{16,15}$ subgraph obtained by deleting the vertex of degree 5 has $196 - 11 = 185$ non-edges, but since $z(16, 15; 5) \leq 184$ (see Table C.6), there must be a $K_{5,5}$ subgraph in the bipartite complement. Thus the maximum degree of vertices in a suitable $K_{16,16}$ subgraph is at most 4 (by the pigeonhole principle, it must be exactly 4). The same argument can be used for the lower bound of some of the larger cases as follows. Suppose in the 67 edge case that we have a vertex of degree 3, then there are 64 edges in the $K_{15,16}$ subgraph obtained by deleting that vertex, but $z(16, 15; 2) = 63$ [Guy69] (Table C.0), so a $K_{2,2}$ must exist. Thus the minimum degree in this case is 4.

The second argument is a more complex counting argument. Suppose that we have a vertex v of degree 6 in the right partition (we do not need to specify the number of edges for this argument). There are 10 vertices in the left partition which do not share an edge with v, call this set C. There are $\binom{10}{2} = 45$ 2-set subsets of C. The most efficient distribution of 2-sets to edges in the right partition (excluding v) is 3 sets per vertex. Since 3 edges make 3 2-sets in the neighborhood of each such vertex there can be at most $3 \times 15 = 45$ edges incident to C. We can add at most one edge from each vertex from the vertices in the right partition (other than v) into v's neighborhood, for 15 more edges. Finally we add the 6 edges incident to v for a total of $45 + 15 + 6 = 66$ edges. Thus we cannot have more than 66 edges and a vertex of degree 6, so in the case of the 67 edges, our maximum degree is at most 5 (by the pigeonhole principle it is exactly 5). If we use the same argument supposing a vertex of degree 7, we determine that at most 62 edges are possible.

The full results of both arguments are summarized in Table 7.0. Despite these bounds, only the $E(G) = 67$ case was computationally feasible. Each of the three graphs generated by nauty for this case contained a $K_{5,5}$ subgraph in the bipartite complement, so none would have been extendable. For the other cases, either better bounds, additional restrictions, or more computing power is needed.

7.2 Heuristic Search to Solve $b(2, 5)$

Using the only known witness for $16 < b(2, 5)$ as a starting point, a heuristic search was attempted. The method used was was to randomly delete edges and vertices and then add back an equal number of vertices and edges generated randomly. This method did not produce any additional witnesses for $16 < b(2, 5)$.

$E(G)$	$\delta(G)$	$\Delta(G)$
60		4
61		≤ 5
62		≤ 6
63		≤ 6
64	≥ 1	≤ 6
65	≥ 2	≤ 6
66	≥ 3	≤ 6
67	4	5

Table 7.0: The conditions on the degrees of the vertices in $K_{16,16}$ subgraphs which might theoretically be extendable to $K_{17,17}$ subgraphs that avoid $K_{2,2}$ and avoid $K_{5,5}$ in the bipartite complement.

7.3 Future Work

These are a list of concrete goals that could be used as the starting point for future research projects.

- Study the Zarankiewicz bounding technique to determine criterion for when the bounds are tight and when they are not. Attempt to improve the theoretical bounds when they are not tight. Further study the error that accumulates due to the occasions when the bound is not tight.

- Attempt to find more additional bounds on Zarankiewicz numbers. A nontrivial bound for extending lower bounds would be useful. An additional approach could involve attempting to bound $z(m, n; s + 1, t)$ given $z(m, n; s, t)$. If such a bound could be found, it might explain the cases where the current theorem's bound is not tight.

- Perform a more thorough heuristic search of $K_{17,17}$ subgraphs in an attempt to further improve the lower bound of $b(2, 5)$ if possible.

- Search for better witnesses to further improve the $b(2, 2, 3)$ lower bound using non-cyclic graphs.

A Acknowledgements

Some of the early work on this project was done during the summer NSF-REU in Extremal Graph Theory and Dynamical Systems 2013 hosted by RIT under Dr. Narayan with the collaboration of Ethan Mark of UC Berkeley. In particular, Ethan helped with the current improvements on $b(2,5)$. Additionally the unsuccessful effort to further improve the lower bound heuristically (as discussed in Section 7.2) should be attributed to him.

The author would like to thank his advisors, Professors Radziszowski, Narayan, and Shahmohamad for all of their support and advice throughout this project.

Additional thanks are owed to Dr. Derrick Stolee of Iowa State for providing advice on undocumented features of nauty.

B Graphs

B.0 Witness for $16 < b(2,5)$

Figure B.0 shows the single witness showing that $16 < b(2,5)$. Table B.0 shows the bipartite adjacency matrix.

1	1	1	1												
				1	1	1	1								
								1	1	1	1				
												1	1	1	1
1				1				1				1			
	1				1				1				1		
		1				1				1				1	
			1				1				1				1
			1			1			1			1			
		1					1	1					1		
	1			1							1			1	
1					1					1					1
	1						1			1		1			
1						1					1		1		
			1		1			1						1	
		1		1					1						1

Table B.0: The bipartite adjacency matrix for the $b(2,5) > 16$ witness.

B.1 Cyclic Witnesses for $15 < b(2,2,3)$

Table B.1 shows the 26 trinary sequences that correspond to the unique witness graphs for $15 < b(2,2,3)$. To arrive at a witness graph, simply construct the bipartite adjacency matrix using each of the 15 rotations of the chosen sequence.

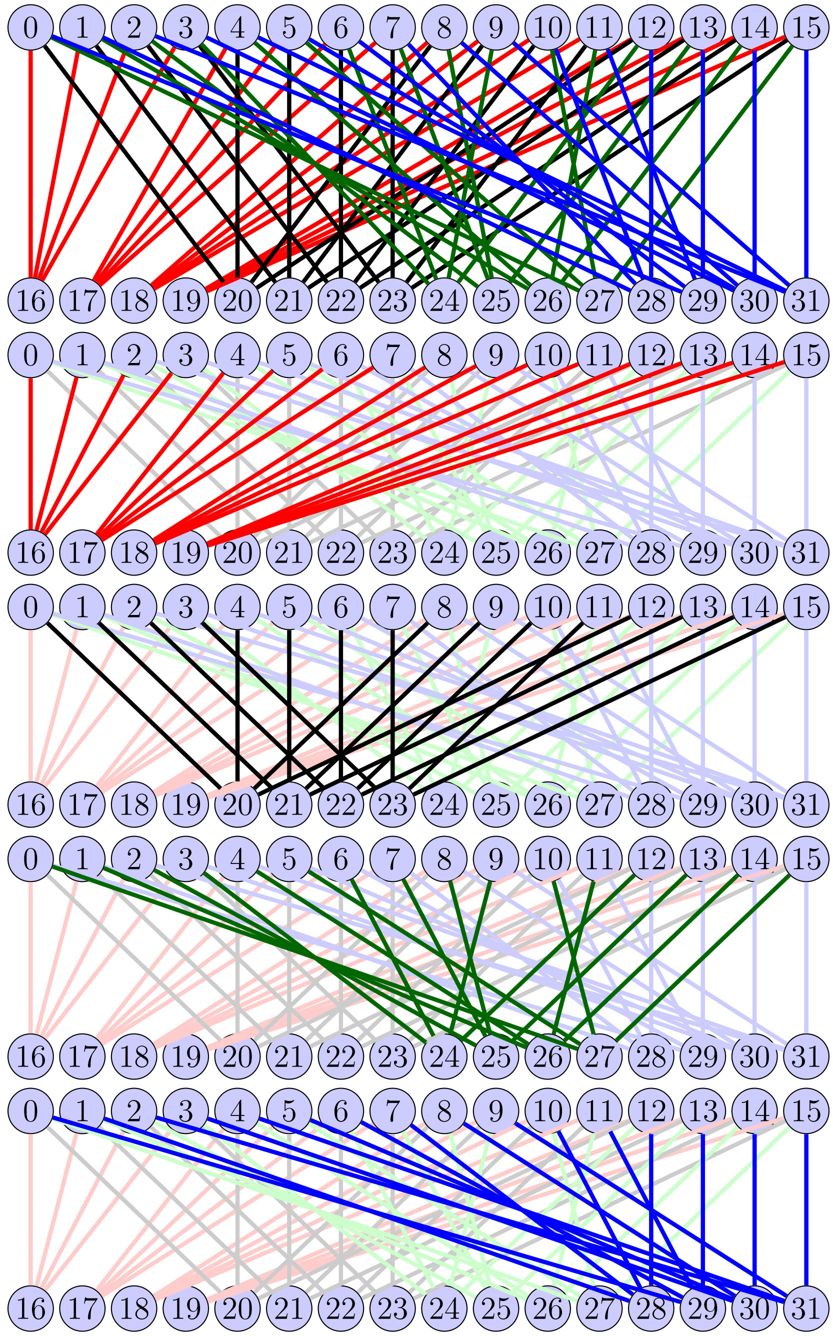

Figure B.0: The graph witnessing $b(2,5) > 16$. First the full graph is shown, then four subgraphs, as color-coded in the original, so that the structure of the graph can be seen more-easily. Clearly there are no C_4s in the graph. In addition, there are no $K_{5,5}$s in the bipartite complement.

0 0 1 0 1 2 2 0 2 2 2 2 1 1 2	0 0 1 2 1 0 2 0 2 2 2 2 1 1 2
0 0 1 0 2 2 1 0 2 1 2 1 2 2 2	0 0 1 2 1 0 2 1 1 2 2 2 2 0 2
0 0 1 0 2 2 1 0 2 2 2 2 1 2 1	0 0 1 2 1 1 0 2 2 2 1 2 2 0 2
0 0 1 0 2 2 1 1 2 1 2 0 2 2 2	0 0 1 2 1 1 2 0 2 2 2 2 0 1 2
0 0 1 0 2 2 1 2 1 1 2 0 2 2 2	0 0 1 2 1 1 2 2 2 0 1 2 2 0 2
0 0 1 0 2 2 1 2 2 2 2 0 1 2 1	0 0 1 2 1 1 2 2 2 2 0 2 0 1 2
0 0 1 0 2 2 2 0 2 1 2 1 1 2 2	0 0 1 2 1 2 2 1 1 2 0 2 0 2 2
0 0 1 1 0 2 1 2 2 0 2 1 2 2 2	0 0 2 0 1 2 1 0 2 2 1 1 2 2 2
0 0 1 1 2 1 2 0 2 1 2 2 0 2 2	0 0 2 0 1 2 1 1 2 2 0 1 2 2 2
0 0 1 1 2 1 2 2 2 0 2 0 1 2 2	0 0 2 0 2 1 2 0 2 1 1 2 1 2 2
0 0 1 1 2 1 2 2 2 1 0 2 0 2 2	0 0 2 1 2 0 2 0 2 1 1 2 2 1 2
0 0 1 1 2 2 1 2 1 0 2 0 2 2 2	0 0 2 1 2 2 2 1 2 0 1 0 1 2 2
0 0 1 2 0 2 2 2 2 0 1 2 1 1 2	0 1 0 1 2 0 1 2 2 0 1 2 2 2 2

Table B.1: The 26 trinary sequences representing the $15 < b(2,2,3)$ witness graphs.

C Zarankiewicz Tables

C.0 Guy's Tables

Table C.0 is a version of Guy's $z(m,n;2)$ table. Table C.1 is the same for $z(m,n;3)$. Table C.2 is for $z(m,n;4)$. Each is slightly modified from Guy's original tables, as explained in each caption. Guy also included off-diagonal tables ($s \neq t$), but they are not included here. [Guy69]

C.1 Extended Tables

Table C.3 is an extended version of the Guy table for $z(m,n;2)$. Table C.4 is the same for $z(m,n;3)$, and Table C.5 is for $z(m,n;4)$. Table C.6 is a new table of $z(m,n;5)$ values. Table C.7 is a new table of $z(m,n;6)$ values. All exact values were checked using nauty, while bounds are from the new theorem introduced in Section 3.0 except where noted.

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3		6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
4			9	10	12	13	14	15	16	17	18	19	20	21	22	23
5				12	14	15	17	18	20	21	22	23	24	25	26	27
6					16	18	19	21	22	24	25	17	28	30	31	32
7						21	22	24	25	27	28	30	31	33	34	36
8							24	26	28	30	32	33	35	36	38	39
9								29	31	33	36	37	39	40	42	43
10									34	36	39	40	42	44	46	47
11										39	42	44	45	47	50	51
12											45	48	49	51	53	55
13												52	53	55	57	59
14													56	58	60	63
15														**61**	63	66

Table C.0: Guy's table of Zarankiewicz numbers $z(m, n; 2)$ [Guy69]. Note that in Guy's original table, he defined Zarankiewicz numbers as the minimum number of edges in a $K_{m,n}$ subgraph to guarantee a $K_{s,t}$ subgraph, which is equivalent, but all of his numbers are increased by one. The numbers here are according to the definition given at the beginning of this paper. Note also that Guy listed $z(15; 2) = 60$. This was shown to be a slight miscalculation; $z(16; 2)$ was recently shown to be 61 [DybET13]. Given that Guy was working with pen and paper in the 1960's, this small error is excusable. The correct value is given. Finally, note that Guy's original table extended 11 more columns to the right, but we have stopped at 17 for brevity.

	3	4	5	6	7	8	9	10	11	12	13
3	8	10	12	14	16	18	20	22	24	26	28
4		13	16	18	21	24	26	28	30	32	34
5			20	22	25	28	30	33	36	38	41
6				26	29	32	36	39	42	45	48
7					33	37	40	44	47	50	53
8						42	45	50	53	57	60
9							49	54	59	64	
10								60			

Table C.1: Guy's table of Zarankiewicz numbers $z(m, n; 3)$ [Guy69]. Note that in Guy's original table, he defined Zarankiewicz numbers as the minimum number of edges in a $K_{m,n}$ subgraph to guarantee a $K_{s,t}$ subgraph, which is equivalent, but all of his numbers are increased by one. The numbers here are according to the definition given at the beginning of this paper. Note also that Guy's original table extends out 10 more columns to the right. This is a duplicate of Table 3.0.

	4	5	6	7	8	9	10	11
4	15	18	21	24	27	30	33	36
5		22	26	30	33	37	41	45
6			31	36	39	43	47	51
7				42	45	49	54	58
8					51	55	60	

Table C.2: Guy's table of Zarankiewicz numbers $z(m, n; 4)$ [Guy69]. Note that in Guy's original table, he defined Zarankiewicz numbers as the minimum number of edges in a $K_{m,n}$ subgraph to guarantee a $K_{s,t}$ subgraph, which is equivalent, but all of his numbers are increased by one. The numbers here are according to the definition given at the beginning of this paper. Note also that Guy's original table extends out 10 more columns to the right.

	13	14	15	16	17	18	19	20	21	22
13	**52**	**53**	**55**	**57**	**59**	61	64	67	69	71
14		**56**	**58**	**60**	**63**	65	68	71	74	76
15			**61**	**63**	**66**	69	72	75	78	81
16				**67**	70	73	76	80	83	86
17					**74**	77	80	84	88	91
18						**81**	84	88	92	96
19							**88**	92	96	100
20								**96**	100	104
21									**105**	109
22										114

Table C.3: An extended table of the Zarankiewicz numbers $z(m, n; 2)$. Exact values are shown in bold, all other values are upper bounds. Off-diagonal exact values were checked using nauty. Those on the diagonal come from Dybizbański, Dzido, and Radziszowski [DybET13]. Upper bounds were generated using the theorem in Section 3.0.

	6	7	8	9	10	11	12	13	14	15
6	**26**	**29**	**32**	**36**	39	42	45	48	51	54
7		33	37	41	45	49	52	56	59	63
8			42	46	51	56	59	63	67	71
9				51	56	61	66	70	75	79
10					62	67	73	77	82	87
11						73	79	84	90	95
12							86	91	98	103
13								98	105	111
14									113	119
15										127

Table C.4: An extended table of the Zarankiewicz numbers $z(m, n; 3)$. Exact values are shown in bold, all other numbers are upper bounds. While related to Guy's table (see Table C.1) this does not rely on Guy for accuracy; all exact values were independently checked using nauty. Upper bounds were generated using the theorem in Section 3.0. This is a duplicate of Table 3.1.

	6	7	8	9	10	11	12	13	14	15
6	**31**	**36**	**39**	**43**	**47**	51	55	59	63	67
7		**42**	**45**	**49**	54	59	64	68	73	78
8			**51**	56	61	67	73	77	82	87
9				63	68	74	80	86	92	97
10					75	82	88	95	102	107
11						90	96	104	112	117
12							104	112	120	127
13								121	130	137
14									140	147
15										157

Table C.5: An extended version of the Zarankiewicz table for $z(m, n; 4)$. Exact values are shown in bold, all other values are upper bounds. While related to Table C.2 (Guy's table), it is not based on this table for accuracy; all exact values were independently checked using nauty. Upper bounds were generated using the theorem in Section 3.0.

	8	9	10	11	12	13	14	15	16	17	18
6	**43**	**48**	**52**	**57**	**62**	**67**	**72**	**76**	**81**	**86**	**91**
7	**50**	**56**	**60**	**66**	**72**	**78**	**84**	88	93	98	103
8	**57**	**64**	**68**	**74**	**80**	**86**	92	98	104	110	116
9		**72**	**76**	**82**	89	96	103	110	117	123	130
10			**84**	91	98	106	111	118	125	132	139
11				100	107	115	122	129	137	145	152
12					116	125	133	140	149	158	165
13						135	144	151	161	171	178
14							155	162	172	182	191
15								173	184	195	204
16									196	208	217
17										221	230
18											243

Table C.6: An extended table of the Zarankiewicz numbers $z(m,n;5)$. Exact values are shown in bold and were checked using nauty. All other values are upper bounds that were generated using the theorem given in Section 3.0, except in the case of $z(10,14;5) \leq 111$ which was determined using a case-by-case argument. If only the bounding technique had been used, $z(10,14;5)$ could only be bounded above by 114, a testament to the innacurracy of the new bounding method.

	6	7	8	9	10	11	12	13	14	15	16	17
6	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**	**80**	**85**	**90**
7		**46**	**52**	**58**	**64**	**70**	**75**	**81**	**87**	**93**	**99**	**105**
8			**59**	**66**	**73**	**80**	**85**	**92**	**99**	**106**	**113**	**120**
9				**74**	**82**	**90**	**95**	**102**	**109**	**116**	123	130
10					**91**	**100**	**105**	**112**	120	128	136	144
11						**110**	**115**	123	132	140	149	158
12							**125**	134	144	152	162	172
13								145	156	164	174	184
14									168	176	187	198
15										188	200	212
16											213	226
17												240

Table C.7: An extended table of the Zarankiewicz numbers $z(m,n;6)$. Exact values are shown in bold and were checked using nauty. All other values are upper bounds that were generated using the theorem given in Section 3.0.

D References

[AlmMan12] J.F. Alm and J. Manske, A New Approach to the Results of Kövari, Sós, and Turán Concerning Rectangle-Free Subsets of the Grid, *Integers*, **12** (2012).

[BalbET07] C. Balbuena, P. García-Vázquez, X. Marcote, and J.C. Valenzuela, New Results on the Zarankiewicz Problem, *Discrete Mathematics*, **307** (2007) 2322-2327.

[BllPpe12] S. Ball and V. Pepe, Asymptotic Improvements to the Lower Bound of Certain Bipartite Turán Numbers, *Combinatorics, Probability & Computing* **21.3** (2012) 323-329.

[BnkSwk75] L. W. Beineke and A. J. Schwenk, On a Bipartite Form of the Ramsey Problem, *Proceedings of the Fifth British Combinatorial Conference, Aberdeen* (1975) 17-22.

[Bol95] B. Bollobás, Extremal Graph Theory, *Handbook of Combinatorics*, Vol. II, Elsevier, Amsterdam 1995, 1231-1292.

[Bwn66] W. G. Brown, On Graphs that do not Contain a Thomsen Graph, *Canadian Mathematical Bulletin*, **9** (1966) 281-285.

[Cnl08] D. Conlon, A New Upper Bound for the Bipartite Ramsey Problem, *Journal of Graph Theory*, **58.4** (2008) 351-356.

[DutRdh12] C. Dutta and J. Radhakrishnan, More on a Problem of Zarankiewicz, *Arxiv-eprints* (2012) 1-14.

[DybET13] J. Dybizbański, T. Dzido and S. Radziszowski, On Some Zarankiewicz Numbers and Bipartite Ramsey Numbers for Quadrilateral, *arXiv preprint arXiv:1303.5475* (2013).

[EReSos66] P. Erdős, A. Rényi and V. T. Sós, On a Problem of Graph Theory, *Studia Scientiarum Mathematicarum Hungarica*, **1** (1966) 215-235.

[Exo91] G. Exoo, A Bipartite Ramsey Number, *Graphs and Combinatorics*, **7** (1991) 395-396.

[FnrET10] S. Fenner, W. Gasarch, C. Glover and S. Purewal, Rectangle Free Coloring of Grids, *arXiv preprint arXiv:1005.3750* (2010).

[FrdSim13] Z. Füredi and M. Simonovits, The History of Degenerate (Bipartite) Extremal Graph Problems, *Erdős Centennial*, (2013) 169-264.

[Gsr98] W. Gasarch, A Survey of Recursive Combinatorics, *Handbook of Recursive Mathematics*, **2**, Elsevier, (1998) 1041-1176.

[GdrET00] W. Goddard, M. A. Henning and O. R. Oellermann, Bipartite Ramsey Numbers and Zarankiewicz Numbers, *Discrete Mathematics*, **219** (2000) 85-95.

[Guy69] R. K. Guy, A Many-Facetted Problem of Zarankiewicz, *Lecture Notes in Mathematics*, **110** (1969) 129-148.

[HttHnn98] J. H. Hattingh and M. A. Henning, Bipartite Ramsey Theory, *Utilitas Mathematica*, **53** (1998) 217-230.

[Hgr13] T. Héger, Some Graph Theoretic Aspects of Finite Geometries, (2013).

[Irv78] R. W. Irving, A Bipartite Ramsey Problem and the Zarankiewicz Numbers, *Glasgow Math*, **19** (1978) 13-26.

[KvrET54] T. Kövári, V. T. Sós, and P. Turán, On a Problem of K. Zarankiewicz, *Colloquium Mathematicum*, **3** (1954) 50-57.

[LzbMub02] F. Lazebnik and D. Mubayi, New Lower Bounds for Ramsey Numbers of Graphs and Hypergraphs, *Advances in Applied Mathematics*, **28** (2002) 544-559.

[LzbWld00] F. Lazebnik and A. Woldar, new Lower Bounds on the Multicolor Ramsey Numbers $r_k(4)$, *Journal of Combinatorial Theory, Series B*, **79** (2000) 172-176.

[LinLi09] Q. Lin and Y. Li, Bipartite Ramsey Numbers Involving Large $K_{n,n}$, *European Journal of Combinatorics*, **30** (2009) 923-928.

[MKy13] B. D. McKay, *nauty 2.5*, `http://cs.anu.edu.au/~bdm/nauty`.

[Nkr10] V. Nikiforov, A Contribution to the Zarankiewicz Problem, *Linear Algrebra and its Applications*, **432** (2010) 1405-1411.

[Prs76] T. D. Parsons, Graphs from Projective Planes, *Aequationes Math*, **14** (1976) 167-189.

[Rms30] F. P. Ramsey, On a Problem of Formal Logic, *Proceedings of the London Mathematical Society*, **30** (1930) 264-286.

[Rmn58] I. Reiman, Über ein Problem von K. Zarankiewicz, *Acta Mathematica Academiae Scientarum Hungaricae*, **9** (1958) 269-273.

[StbPhf12] B. Steinbach and C. Posthoff, Extremely Complex 4-Colored Rectangle-Free Grids: Solution of Open Multiple-Valued Problems, *Multiple-Valued Logic (ISMVL), 2012 42nd IEEE International Symposium on,* (2012) 37-44.

[Wrn12] M. Werner, An Algorithmic Approach for the Zarankiewicz Problem, (2012).

[Zkz51] K. Zarankiewicz, Problem p 101 (French), *Colloq. Math.* **2** (1951) 301.